# Graded Bourbaki ideals of graded modules and Ideals of reduction number two

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This talk is based on the following papers:

References


- [J. Herzog, D. I. Stamate, K], **Graded Bourbaki ideals of graded modules**, Math. Z., 299, 1303–1330, 2021
- [K], **Ideals of reduction number two**, Israel J. Math., 243, 45–61, 2021
- [K], **Graded filtrations and ideals of reduction number two**, Math. Nachr., (to appear).

# Graded Bourbaki ideals of graded modules

# Introduction

**Fact [Bourbaki]**

Let $R$ be a Noetherian normal domain and
$M$ be a (f.g.) torsionfree $R$-module of rank $r > 0$. Then,

$$\exists 0 \to R^{r-1} \to M \to I \to 0,$$

where $I$ is a nonzero ideal of $R$.

**Philosophy**

Properties of a module are inherited by those of its Bourbaki ideals.

Ex.

- the vanishing of cohomologies
- the study of the maximal Cohen-Macaulay modules over hypersurface rings
- the Rees algebras of modules
- the Hilbert function
- $\cdots$

## Observation

Let $R = K[X, Y, Z]$ and $M = \left\langle \begin{pmatrix} 0 \\ -Z \\ Y \end{pmatrix}, \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix}, \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix} \right\rangle \subseteq R^3$.
Then, $M$ is a torsionfree $R$-module of rank 2 and

$$\exists 0 \to R \longrightarrow M \to (Y, Z) \to 0,$$

$$\exists 0 \to R \longrightarrow M \to (X, Z) \to 0.$$

**Question**

- How to find a Bourbaki sequence?
- How many Bourbaki sequences are there?

In what follows, let $R$ be a Noetherian normal domain, $M$ be a f.g. $R$-module of rank $r > 0$.

# Criteria to be a Bourbaki sequence

## Fact

The following hold true:

- $M$ is torsionfree $\Leftrightarrow \exists 0 \to M \to R^s$.
- $M$ is reflexive $\Leftrightarrow \exists 0 \to M \to R^s \to R^t$.

## Theorem [Herzog-Stamate-K]

Suppose that $M$ is reflexive and choose $0 \to M \xrightarrow{\iota} R^s \to R^t$. Then, for a homomorphism $\varphi : R^{r-1} \to M$ of modules, the following are equivalent:

- $0 \to R^{r-1} \xrightarrow{\varphi} M \to \mathrm{Coker}\varphi \to 0$ is a Bourbaki sequence.
- $\mathrm{ht}_R(I_{r-1}(\iota \circ \varphi)) \geq 2$.

## Example

Let $R = K[X, Y, Z]$ and $M = \left\langle \begin{pmatrix} 0 \\ -Z \\ Y \end{pmatrix}, \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix}, \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix} \right\rangle \overset{\iota}{\subseteq} R^3$.
Then, $M$ is reflexive since $M = \Omega_R^2(K)$. Let

$$\varphi : R \to M; 1 \mapsto f \cdot \begin{pmatrix} 0 \\ -Z \\ Y \end{pmatrix} + g \cdot \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix} + h \cdot \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix}.$$

Then, $\iota \circ \varphi : R \to M \to R^3; 1 \mapsto \begin{pmatrix} -Zg - Yh \\ -Zf + Xh \\ Yf + Xg \end{pmatrix}$.
Hence,

$$\begin{aligned} & 0 \to R \xrightarrow{\varphi} M \to \mathrm{Coker}\varphi \to 0 \text{ is a Bourbaki sequence} \\ \Leftrightarrow \quad & \mathrm{ht}_R I_1 \begin{pmatrix} -Zg - Yh \\ -Zf + Xh \\ Yf + Xg \end{pmatrix} \geq 2. \end{aligned}$$

## Example - continuation

Thus,

$$\gcd I_1 \left( \begin{smallmatrix} -Zg-Yh \\ -Zf+Xh \\ Yf+Xg \end{smallmatrix} \right) \begin{cases} = 1 & \text{if } (f,g,h) = (1,0,0),\ (0,1,0)... \\ \neq 1 & \text{if } (f,g,h) = (X,0,0), (Z,Y,X)... \end{cases}$$

# Ubiquity of graded Bourbaki sequences

**Fact (graded version of Bourbaki's theorem)**

Let

- $R = \bigoplus_{n \geq 0} R_n$ be a standard graded Noetherian normal domain of dimension $\geq 2$ s.t. $R_0$ is an infinite field,
- $M$ be a graded torsionfree $R$-module of rank $r > 0$, and
- $k \geq \max\{\deg f : f \in M \text{ is a graded min. gen. of } M\}$.

Then,

$$\exists 0 \to R(-k)^{r-1} \to M \to I(m) \to 0,$$

where $I$ is a graded ideal of $R$ and $m \in \mathbb{Z}$.

Under the assumptions of Fact, for arbitrary graded homomorphism $\varphi : R(-k)^{r-1} \to M$ of modules, we have the commutative diagram

$$\begin{tikzcd}
 & G = R(-k)^{\alpha} \arrow[d, "\pi\text{:surj.}"] \\
 & RM_k \arrow[d, "\iota\text{:inj.}"] \\
F = R(-k)^{r-1} \arrow[r, "\forall\varphi"] \arrow[uur, dotted, "\exists A_{\varphi}"] & M.
\end{tikzcd}$$

With the above notation, we obtain the following.

## Theorem [Herzog-Stamate-K]

In addition to the assumption of Fact, suppose that

- $R$ is a CM ring s.t. $K = R_0$ is an alg. closed field and
- $M$ is reflexive.

For fixed free basis $F$ and $G$, let $A \in K^{\alpha \times (r-1)}$ denote the matrix representing $F \to G$. Then,

$$\left\{ A \in K^{\alpha \times (r-1)} : \begin{array}{c} 0 \to F \xrightarrow{\iota \circ \pi \circ A} M \to \mathrm{Coker} \to 0 \\ \text{is a Bourbaki sequence} \end{array} \right\}$$

is a nonempty Zariski open subset of $K^{\alpha \times (r-1)}$.

## Example

Let $R = K[X, Y, Z]$ with $\deg X = \deg Y = \deg Z = 1$ and $M = \left\langle \begin{pmatrix} 0 \\ -Z \\ Y \end{pmatrix}, \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix}, \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix} \right\rangle \subseteq R^3$. Then, $M$ is generated in degree 2. For $a, b, c \in K$, set

$$\varphi_{(a,b,c)} : R(-2) \to M; 1 \mapsto a\cdot \begin{pmatrix} 0 \\ -Z \\ Y \end{pmatrix} + b\cdot \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix} + c\cdot \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix}.$$

Then,

$$\left\{ (a, b, c) \in K^3 : \underset{\text{is a Bourbaki sequence}}{0 \to F \xrightarrow{\varphi_{(a,b,c)}} M \to \mathrm{Coker} \to 0} \right\} = K^3 \setminus \{(0, 0, 0)\}.$$

# Ideals of reduction number two

# Introduction

Let

- $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d$ and
- $I$ an $\mathfrak{m}$-primary ideal.

Then $\ell_A(A/I^{n+1})$ agrees with a polynomial function for $n \gg 0$, i.e. there exist integers $\mathrm{e}_0(I), \mathrm{e}_1(I), \ldots, \mathrm{e}_d(I)$ such that

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

for all $n \gg 0$.

### Philosophy

The Hilbert function $\ell_A(A/I^{n+1})$ reflects the structures of

- the **Rees algebra** $\mathcal{R}(I) = A[It] = \bigoplus_{n \geq 0} I^n t^n$ and
- the **associated graded ring** $\mathcal{G}(I) = \mathcal{R}(I)/I\mathcal{R}(I) = \bigoplus_{n \geq 0} (I^n/I^{n+1}) t^n$.

Remark: $\ell_A(A/I^{n+1}) - \ell_A(A/I^n) = \ell_A(\mathcal{G}(I)_n)$.

In what follows, let

- $(A, \mathfrak{m})$ be a CM local ring of dimension $d \geq 2$,
- $I$ an $\mathfrak{m}$-primary ideal, and
- $A/\mathfrak{m}$ an infinite field.

Choose a parameter reduction $Q (\subseteq I)$ of $I$, i.e., $I^{n+1} = QI^n$ for some $n \geq 0$. Set the **reduction number** as

$$\mathrm{red}_Q I = \min\{n \geq 0 \mid I^{n+1} = QI^n\}.$$

**Fact [Rees, Northcott, Huneke, Ooishi]**

- $\mathrm{red}_Q I = 0 \quad \Rightarrow \quad \mathcal{G}(I) \cong (A/I)[X_1, \ldots, X_d]$.
- In general, $\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I)$ holds, and $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I)$ if and only if $\mathrm{red}_Q I = 1$. When this is the case, $\mathcal{G}(I)$ is a CM ring.

**Question**

$$\mathrm{red}_Q I = 2 \quad \Rightarrow \quad ???$$

Note that

- $\exists$ parameter reductions $Q_1$ and $Q_2$ of $I$ such that $\mathrm{red}_{Q_1} I = 2$ and $\mathrm{red}_{Q_2} I = 3$ ([Marley, 1993]).
- $\exists I$ with $\mathrm{red}_Q I = 2$ such that $\operatorname{depth} \mathcal{G}(I) = 0$.

**Theorem [K, Israel J.]**

$I^3 = QI^2$ and $\mathfrak{m}I^2 \subseteq QI \quad \Rightarrow \quad \ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$.
"$=$" holds if and only if $\operatorname{depth} \mathcal{G}(I) \geq d - 1$.

**Question**

$$\mathrm{red}_Q I = 2 \quad \Rightarrow \quad ???$$

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- $\exists$ parameter reductions $Q_1$ and $Q_2$ of $I$ such that $\mathrm{red}_{Q_1} I = 2$ and $\mathrm{red}_{Q_2} I = 3$ ([Marley, 1993]).
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A graded $\mathcal{R}(Q)$-module

$$S = I\mathcal{R}(I)/I\mathcal{R}(Q) = \bigoplus_{n \geq 0} (I^{n+1}/Q^n I)t^n$$

is called the **Sally module** of $I$ w.r.t. $Q$.

## Fact

- $\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - (\mathrm{e}_0(I) - \ell_A(A/I))\binom{n+d-1}{d-1} - \ell_A(S_n)$ for all $n \geq 0$.
- $\mathfrak{m}^\ell S = 0$ for $\ell \gg 0$.
- If $S \neq 0$, then $\operatorname{Ass}_{\mathcal{R}(Q)} S = \{\mathfrak{m}\mathcal{R}(Q)\}$.
- $\operatorname{depth} \mathcal{G}(I) \geq d-1 \Leftrightarrow S$ is either 0 or a CM $\mathcal{R}(Q)$-module.

Idea of the proof:

- By Fact, $S$ is a torsionfree $\mathcal{R}(Q)/\mathfrak{m}^{\ell}\mathcal{R}(Q)$-module for $\ell \gg 0$.
- The assumptions $I^3 = QI^2$ and $\mathfrak{m}I^2 \subseteq QI$ show that $\ell = 1$.
- $\exists 0 \to P(-1)^{r-1} \to S \to I(m) \to 0$, where $P = \mathcal{R}(Q)/\mathfrak{m}\mathcal{R}(Q) \cong (A/\mathfrak{m})[X_1, \dots, X_d]$.

# Further results

By constructing another filtration, we can remove the assumption that $\mathfrak{m}I^2 \subseteq QI$:

**Theorem [K, Math. Nachr.]**

$\mathrm{red}_Q I = 2 \quad \Rightarrow \quad \ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$.
"$=$" holds if and only if $\operatorname{depth} \mathcal{G}(I) \geq d - 1$.

Thank you for the attention!