# The Auslander-Reiten conjecture for non-Gorenstein rings

**Shinya Kumashiro**

Chiba University

The 8th China-Japan-Korea International Symposium on Ring Theory

August 30th, 2019

# Introduction

In what follows, let

- $R$ be a commutative ring and
- $M$ a finitely generated $R$-module.

**Conjecture 1 (Auslander-Reiten, 1975).**

(ARC) $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0 \Rightarrow M$ is projective.

# Introduction

In what follows, let

- $R$ be a commutative ring and
- $M$ a finitely generated $R$-module.

**Conjecture 1 (Auslander-Reiten, 1975).**

(ARC) $\quad \mathrm{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0 \Rightarrow M$ is projective.

# Known results

Let $R$ be a Noetherian local ring.
Then **(ARC)** holds for $R$ if $R$ satisfies one of the following.

- Gorenstein normal domain. [Araya]
- complete intersection. [Auslander-Ding-Solberg]
- Gorenstein s.t. $\mathrm{e}(R) - \dim R \leq 4$. [Şega]
- CM normal excellent $\mathbb{Q}$-algebra. [Huneke-Leuschke]
- Golod. [Jorgensen-Şega]

# Main results

Let $s \leq t$ be positive integers and $A$ a commutative ring. Set

- $A[\mathbf{X}] = A[X_{ij}]_{1 \leq i \leq s, 1 \leq j \leq t}$ **:** polynomial ring over $A$
- $\mathbb{I}_s(\mathbf{X})$ **:** ideal of $A[\mathbf{X}]$ generated by the maximal minors of the matrix $(X_{ij})$.

With these notations, we have the following.

**Theorem A**

Suppose $A$ is either a complete intersection or a Gorenstein normal domain. Then **(ARC)** holds for the determinantal ring $A[\mathbf{X}]/\mathbb{I}_s(\mathbf{X})$ if $2s \leq t + 1$.

# Main results

Let $(R, \mathfrak{m})$ be a CM local ring and $I$ an $\mathfrak{m}$-primary ideal. Then $I$ is an *Ulrich ideal* if

- $I$ is not a parameter ideal, but $I^2 = \mathfrak{q}I$ for some parameter ideal $\mathfrak{q}$.
- $I/I^2$ is a free $R/I$-module.

**Theorem B**

Let $R$ be a CM local ring. Suppose that

$$\exists I\text{: Ulrich ideal s.t. } R/I \text{ is a c.i.}$$

Then **(ARC)** holds for $R$.

# Theorem A

(for determinantal rings)

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $Q = (x_1, x_2, \ldots, x_n)$ an ideal of $R$ generated by a regular sequence on $R$.

**Fact [Auslander-Ding-Solberg]**

$R$ satisfies (ARC) $\Leftrightarrow$ $R/Q$ satisfies (ARC).

**Key proposition [K]**

Supoose that $R$ is Gorenstein. If $0 < \ell \leq n$, then

$$R \text{ satisfies (ARC)} \Leftrightarrow R/Q^\ell \text{ satisfies (ARC).}$$

In particular, $R$ satisfies (ARC) $\Leftrightarrow$ $R/Q^2$ satisfies (ARC).

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $Q = (x_1, x_2, \ldots, x_n)$ an ideal of $R$ generated by a regular sequence on $R$.

**Fact [Auslander-Ding-Solberg]**

$$R \text{ satisfies (ARC)} \Leftrightarrow R/Q \text{ satisfies (ARC)}.$$

**Key proposition [K]**

Supoose that $R$ is Gorenstein. If $0 < \ell \leq n$, then

$$R \text{ satisfies (ARC)} \Leftrightarrow R/Q^\ell \text{ satisfies (ARC)}.$$

In particular, $R$ satisfies (ARC) $\Leftrightarrow$ $R/Q^2$ satisfies (ARC).

**Corollary**

Let $s \leq t$ be positive integers. Suppose that

- $R$ is a Gorenstein local ring and
- $\{x_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq t}$ forms a regular sequence on $R$.
- $\{n_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq t}$ are positive integers.

Then (ARC) holds for $R$ if and only if it holds for $R/\mathbb{I}_s(x_{ij}^{n_{ij}})$, if $2s \leq t+1$.

Suppose that

- $(A, \mathfrak{n})$: Gorenstein normal domain or c.i.
- $Q = (x_1, x_2, \ldots, x_d)$: parameter ideal of $A$.

$*$ d: dimension of $A$.

**Example**

- $A/I$ where
  $I = \mathbb{I}_2 \left( \begin{smallmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{smallmatrix} \right) = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5)$.
- $A[[t^5, t^7, t^8, t^{11}]]$.
- Three generated numerical semigroup rings $A[[t^a, t^b, t^c]]$.
- Rees algebras $A[Qt]_{\mathfrak{m}}$, where $\mathfrak{m} = \mathfrak{n}A[Qt] + A[Qt]_+$.

# Theorem B
(for Ulrich ideals)

In what follows, let $(R, \mathfrak{m})$ be a CM local ring.

**Key proposition [K]**

The following are equivalent.

- $\exists \mathfrak{q}$: a parameter ideal of $R$,
  $\exists S$: a c.i. local ring of positive dimension, and
  $\exists Q$: a parameter ideal of $S$ s.t. $R/\mathfrak{q} \cong S/Q^2$.

- $\exists I$: an $\mathfrak{m}$-primary ideal and
  $\exists \mathfrak{q}$: a parameter ideal of $R$ s.t.
  - $I^2 \subseteq \mathfrak{q} \subsetneq I$ and $I/\mathfrak{q}$ is $R/I$-free.
  - $R/I$ is a complete intersection.

**Definition [Goto-Ozeki-Takahashi-Watanabe-Yoshida]**

For an $\mathfrak{m}$-primary ideal $I$, $I$ is an *Ulrich ideal* if

- $I$ is not a parameter ideal, but $I^2 = \mathfrak{q}I$ for some parameter ideal $\mathfrak{q}$.
- $I/I^2$ is a free $R/I$-module.

Note that the second condition can be replaced that

$$I/\mathfrak{q} \text{ is } R/I\text{-free},$$

because $0 \to \mathfrak{q}/\mathfrak{q}I \to I/I^2 \to I/\mathfrak{q} \to 0$ is exact and $\mathfrak{q}/\mathfrak{q}I \cong R/I \otimes_R \mathfrak{q}/\mathfrak{q}^2$ is $R/I$-free.

**Theorem**

Suppose that

$$\exists I\text{: Ulrich ideal s.t. } R/I \text{ is a c.i.}$$

Then the following assertions are true.

- $d + r \leq v$.
- $\exists \mathfrak{q}$: a parameter ideal of $R$,
  $\exists S$: a c.i. local ring of dimension $r > 0$ and
  $\exists Q$: a parameter ideal of $S$ s.t. $R/\mathfrak{q} \cong S/Q^2$.
- For a f.g. $R$-module $M$,
  $\mathrm{pd}_R\, M = \inf\{n \geq 0 \mid \mathrm{Ext}_R^n(M, M \oplus R) \neq 0\}$.

$*$ d:dimension, r:CM type, v:embedding dimension.

With the same assumptions as in previous Theorem, we further have the following.

## Corollary

Assume that $\exists T \to R$: a surjective ring homomorphism, where $T$ is a regular local ring $T$ of dimension $v$. Let

$$0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0$$

be a minimal $T$-free resolution of $R$. Then

$$\mathrm{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \cdot \binom{v-r-d}{j}$$

for $1 \leq i \leq v-d$, where $\beta_k = \begin{cases} 1 & \text{if } k = 0 \\ k \cdot \binom{r+1}{k+1} & \text{if } 1 \leq k \leq r \\ 0 & \text{otherwise.} \end{cases}$

# Example

Set

- $R = k[[t^6, t^{11}, t^{16}, t^{26}]]$.
- $I = (t^6, t^{16}, t^{26})$: an ideal of $R$.

Then $I$ is an Ulrich ideal of $R$ and $R/I$ is a c.i.
Therefore the minimal $T = k[[X, Y, Z, W]]$-free resolution of $R$ has the following form;

$$0 \to T^{\oplus 2} \to T^{\oplus 5} \to T^{\oplus 4} \to T \to R \to 0.$$

Furthermore,

$$R \cong T/\left(X^7 - ZW, Y^2 - XZ, Z^2 - XW, W^2 - X^6 Z\right).$$

Thank you for your attention.