

The Auslander-Reiten conjecture
for certain non-Gorenstein Cohen-Macaulay rings

WVU Algebra Seminar via Zoom

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§.1 Introduction

Setting

- R : comm. Noeth. ring
- M : f.g. R -mod

Auslander-Reiten conjecture:

$\text{Ext}_R^{\geq 0}(M, M \oplus R) = 0 \Rightarrow M$ is projective.

Note that $\cdot \Leftarrow$ holds true.

◦ \exists counter ex. if R is a non-comm Artinian ring
[Schulze].

On the other hand, there are many affirmative answers,
in comm. (especially higher-dimensional) rings.

Partial results:

The Auslander-Reiten conjecture holds for

- complete intersections.
 - Gorenstein normal domains. [Huneke-Leuschke, Araya]
 - Gorenstein rings with $e(R) - \dim R \leq 4$. [Şega]
 - Cohen-Macaulay normal domain \mathbb{Q} -algebras [HL]
 - Golod rings [Jorgensen-Şega].
- ;

Fundamental fact:

(1) R_m satisfies (ARC) for $\forall m \in \text{Max} R$
 $\Rightarrow R$ satisfies (ARC).

(2) Suppose that

- (R, m) : Noether local ring
- $x \in m$: NZD of R .

Then R satisfies (ARC)

$\Leftrightarrow R/xR$ satisfies (ARC)

- (1) is easy.
- (2) follows from the result of Auslander-Ding-Solberg, which is about the lifting problem.

- By Fact (1), (2), if all Artinian local rings satisfies (ARC), then so do all CM rings!

(Big) Question

Do all CM rings satisfy (ARC)?

(or Gorenstein rings)



Huneke-Wiegand conjecture

Outline of my talk:

1st talk: Fact (2). and the lifting problem.

2nd talk:

Question:

(R, \mathfrak{m}) : Noeth. local ring

\mathfrak{Q} : ideal generated by a reg. seq. of R

Then, for $\ell > 0$, R satisfies (ARC)

$\Leftrightarrow R/\mathfrak{Q}^\ell$ satisfies (ARC)?

- If $l > 1$, R/Q^e is neither a Gorenstein ring nor a domain.
- The deformation R/Q^e relates determinantal rings and the existence of "special" ideals

In what follows, let

- (R, m) : Noeth. local ring.
- $x \in m$: NZD of R
- M : f.g. R -mod.

§2. Proof of Fact (2)

lem

$$(1) \quad 0 \rightarrow \Omega \rightarrow F \rightarrow M \rightarrow 0 \quad (ex) \text{ and} \\ \text{f.g. free}$$

$$\text{Ext}_R^{>0}(M, M \oplus R) = 0 \Rightarrow \text{Ext}_R^{>0}(\Omega, \Omega \oplus R) = 0$$

(2) Let $x \in m$ be a NZD of R and M .

$$\text{Then } \text{Ext}_R^{>0}(M, M \oplus R) = 0$$

$$\Leftrightarrow \text{Ext}_{R/xR}^{>0}(M/xM, M/xM \oplus R/xR) = 0$$

Thm (Fact (2)) Let $x \in M$ be a NZD of R .

Then $R : (\text{ARC}) \Leftrightarrow R/xR : (\text{ARC})$.

Proof) (\Leftarrow) : Let M be a f.g. R -mod. s.t.
 $\text{Ext}_R^{\infty}(M, M \oplus R) = 0$

lem (1)

$$\Rightarrow \text{Ext}_R^{\infty}(\Omega, \Omega \oplus R) = 0$$

lem (2)

$$\Rightarrow \text{Ext}_{R/xR}^{\infty}(R/xR \otimes \Omega, R/xR \otimes \Omega \oplus R/xR) = 0$$

$R/xR : (\text{ARC})$

$$\Rightarrow R/xR \otimes \Omega = R/xR\text{-free}$$

$$\Rightarrow \Omega = R\text{-free}$$

$$\Rightarrow M = R\text{-free}$$

(\Rightarrow) : N : f.g. R/xR -mod s.t. $\text{Ext}_{R/xR}^{\infty}(N, N \oplus R/xR) = 0$

In order to proceed in the same way as above,
we face the following problem:

Problem Is there a f.g. R -mod M

s.t. $\bullet M \otimes_R M \cong N$

- \bullet the min. R -free res. $F_0 \rightarrow M \rightarrow 0$ of M induces

the min. $R \otimes R$ -free res. of N :

$$F_0 \otimes_R F_0 \rightarrow M \otimes_R M \cong N \rightarrow 0?$$

If yes, then

$$0 \rightarrow M \xrightarrow{x} M \rightarrow N \rightarrow 0 \quad (R \otimes R) \quad \downarrow \text{Hom}_R(_, M \otimes R)$$

$$\text{Ext}_R^{2,0}(M, M \otimes R) = 0 \quad \text{by NAK.}$$

$$\begin{matrix} R: (\text{ARC}) \\ \rightsquigarrow \bullet \end{matrix} M: R\text{-free}$$

$$\rightsquigarrow \bullet N: R \otimes R\text{-free}$$



§ 3. Lifting problem

- Let $\bullet R \rightarrow S$: ring hom. of Noeth. rings.
 $\bullet N$: f.g. S -mod.

Then N is liftable to R

$\stackrel{\text{def}}{\iff} \exists M: \text{f.g. } R\text{-mod s.t.}$

$F_0 \rightarrow M \rightarrow 0: R\text{-free res. of } M$
induces an S -free res of N :

$$S \otimes F_0 \rightarrow S \otimes_R M \rightarrow 0 \quad (\otimes)$$

$$\downarrow \cong$$

$$N$$

$\Leftrightarrow \exists M: \text{f.g. } R\text{-mod}$

s.t. $\begin{cases} S \otimes_R M \cong N \text{ and} \\ \text{Tor}_{>0}^R(S, M) = 0 \end{cases}$

Lifting problem

When is an S -mod N liftable to R ?

Thm [Auslander-Ding-Solberg].

- Suppose
- (R, \mathfrak{m}) : Noeth. local ring.
 - $x \in \mathfrak{m}$: R -NZD.
 - N : f.g. R_x/R -mod.

If $\text{Ext}_{R_x/R}^2(N, N) = 0$, then
 N is liftable to R .

Set $R_i := R_x^i R$ for $i \geq 0$.

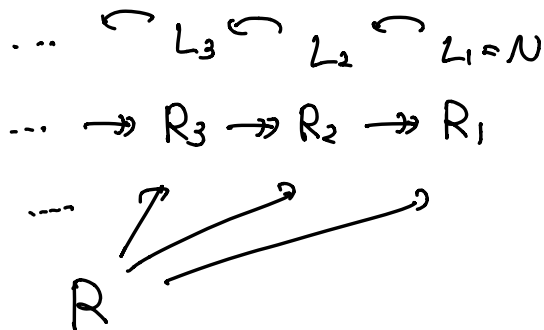
Prop Let N be a fig. R_1 -mod.

If $\exists \{L_i : R_i\text{-mod}\}_{i>0}$.

s.t. $\begin{cases} \cdot L_1 \cong N \\ \cdot L_{i+1} \text{ is a lifting of } L_i, \end{cases}$

then N is liftable to R .

Philosophy)



\rightsquigarrow The R -mod. $\varprojlim_i L_i$ is what we desired.

Proof of Thm)

- Let $i \geq 1$. Suppose
- $N : \text{fig. } R_i\text{-mod}$
 - $L_i : \text{fig. } R_i\text{-mod}$ s.t.
 L_i is a lifting of N .

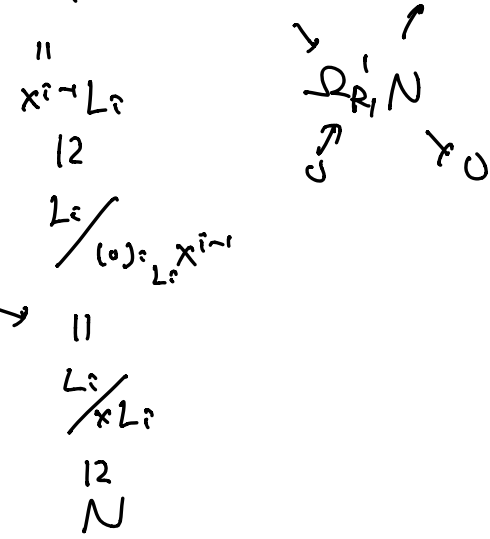
ETS: L_i is liftable to R_i

Consider the min. R -free res. of L_i :

$$0 \rightarrow \Omega \rightarrow F \rightarrow L_i \rightarrow 0$$

$$\rightsquigarrow 0 \rightarrow (0)_{L_i} \rightarrow \frac{R}{x\Omega} \rightarrow \frac{F}{x F} \rightarrow N \rightarrow 0 \text{ ex.}$$

These follow from that L_i is a lifting of N



On the other hand,

$$0 \rightarrow N \rightarrow \frac{R}{x\Omega} \rightarrow \Omega_{R_i}^1 N \rightarrow 0 \text{ ex}$$

$$\begin{aligned} & \updownarrow (10) \\ \exists \quad 0 & \in \text{Ext}_{R_i}^1(\Omega_{R_i}^1 N, N) \\ & \parallel \\ & \text{Ext}_{R_i}^2(N, N) \end{aligned}$$

Claim : $\Omega = 0 \Rightarrow L_i$ is liftable to R_{i+1}

Proof) $\Omega = 0 \Leftrightarrow 0 \rightarrow N \xrightarrow{f} \frac{Q}{\times \Omega} \xrightarrow{g} \frac{Q}{\times \Omega} \rightarrow N \rightarrow 0$
 (i.e. $\begin{array}{ccc} \parallel & \circlearrowleft & \parallel \\ N & \xrightarrow{\beta} & N \end{array}$) splits.

$\Rightarrow 0 \rightarrow \Omega \rightarrow F \rightarrow L_i \rightarrow 0$
 $\begin{array}{c} \varepsilon \downarrow \\ \frac{Q}{\times \Omega} \\ \beta \downarrow \\ 0 \rightarrow N \rightarrow F \rightarrow L_i \rightarrow 0 \end{array} \quad \begin{array}{c} \parallel \\ \text{p.o.} \\ \parallel \end{array}$

Then F is a f.g. R_{i+1} -mod. and a lifting of L_i .

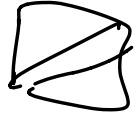
Proof) \leftarrow $R_i \otimes_R -$

$$\begin{array}{ccccccc} 0 & \rightarrow & N & \xrightarrow{f} & \frac{Q}{\times \Omega} & \rightarrow & \frac{Q}{\times \Omega} \rightarrow N \rightarrow 0 \\ & & \parallel & & \beta \downarrow & & \downarrow & \parallel \\ \dots & \rightarrow & N & \xrightarrow{\tau} & N & \rightarrow & \frac{F}{\times E} \rightarrow N \rightarrow 0 \end{array}$$

Hence, since $\beta f = id_N$, τ is surj $\Rightarrow \tau$ is bij.

Hence $\mathbb{F}/\mathbb{E} \cong N$.

(To get $\text{Tor}_{\geq 0}^{R_i} (R_i, \mathbb{E}) = 0$ is another story.)



Corollary

Let R be a complete intersection local ring
and M a f.g. R -mod. Then

$$\text{Ext}_R^2(M, M) = 0 \Rightarrow \text{pd}_R M < \infty.$$

In particular, R satisfies (ARC).