The Auslander-Reiten conjecture for certain non-Gorenstein Cohen-Macaulay rings

WVU Algebra Seminar via Zoom

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§.1 Introduction

Setting
- R : comm. Noeth. ring
- M : f.g. R-mod

Auslander-Reiten conjecture:

$\operatorname{Ext}_R^{>0}(M, M \oplus R) = 0 \Rightarrow M$ is projective.

Note that
- $\Leftarrow$ holds true.
- $\exists$ counter ex. if R is a non-comm. Artinian ring [Schultz].

On the other hand, there are many affirmative answers, in comm. (especially higher-dimensional) rings.

Partial results:

The Auslander-Reiten conjecture holds for

- complete intersections.
- Gorenstein normal domains. [Huneke-Leuschke, Araya]
- Gorenstein rings with $e(R) - \dim R \leq 4$. [Şega]
- Cohen-Macaulay normal domain $\mathbb{Q}$-algebras [HL]
- Golod rings [Jorgensen-Şega].

⋮

Fundamental fact:

(1) $R_{\mathfrak{m}}$ satisfies (ARC) for $\forall \mathfrak{m} \in \operatorname{Max} R$

$\Rightarrow$ R satisfies (ARC).

(2) Suppose that
- $(R, \mathfrak{m})$: Noeth local ring
- $x \in \mathfrak{m}$: NZD of R.

Then R satisfies (ARC)

$\Leftrightarrow$ R/xR satisfies (ARC).

- (1) is easy.
- (2) follows from the result of Auslander-Ding-Solberg, which is about the lifting problem.

∘ By Fact (1), (2), if all Artinian local rings satisfies (ARC), then so do all CM rings!

<u>(Big) Question</u>

Do all CM rings satisfy (ARC)?

(or Gorenstein rings)

$\Updownarrow$

Huneke-Wiegand conjecture.

Outline of my talk:

1st talk: Fact (2). and the lifting problem.

2nd talk:

<u>Question:</u>

$(R, \mathfrak{m})$: Noeth. local ring

Q: ideal generated by a reg. seq. of R

Then, for $\ell > 0$, R satisfies (ARC)

$\Leftrightarrow$ R/Q^{ℓ} satisfies (ARC)?

- If $\ell > 1$, R/Q^ℓ is neither a Gorenstein ring nor a domain.

- The deformation R/Q^ℓ relates determinantal rings and the existence of "special" ideals

In what follows, let
- $(R, \mathfrak{m})$: Noeth. local ring.
- $x \in \mathfrak{m}$: NZD of R
- M : f.g. R-mod.

§2. Proof of Fact (2)

<u>lem</u>

(1) $0 \to \Omega \to \underset{\text{f.g. free}}{F} \to M \to 0$ (ex) and

$$\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0 \Rightarrow \mathrm{Ext}_R^{>0}(\Omega, \Omega \oplus R) = 0$$

(2) Let $x \in \mathfrak{m}$ be a NZD of R and M.

Then $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$

$$\Leftrightarrow \mathrm{Ext}_{R/xR}^{>0}(M/xM, M/xM \oplus R/xR) = 0$$

Proof) (1): $0 \to \Omega \to F \to M \to 0$ ① $\mathrm{Hom}_R(-,R)$ ② $\mathrm{Hom}_R(-,\Omega)$ ③ $\mathrm{Hom}_R(M,-)$

① $\rightsquigarrow \mathrm{Ext}_R^{>0}(\Omega,R)=0$

② $\rightsquigarrow \mathrm{Ext}_R^{i}(\Omega,\Omega) \simeq \mathrm{Ext}_R^{i+1}(M,\Omega)$ for $\forall i>0$.

③ $\rightsquigarrow \mathrm{Ext}_R^{\geq 1}(M,\Omega)=0$

Hence $\mathrm{Ext}_R^{>0}(\Omega,\Omega\oplus R)=0$ //

(2): $0 \to M \xrightarrow{x} M \to M/xM \to 0$ $\mathrm{Hom}_R(-,M\oplus R)$

$$\cdots \to \mathrm{Ext}_R^i(M/xM, M\oplus R) \to \mathrm{Ext}_R^i(M, M\oplus R) \xrightarrow{x} \mathrm{Ext}_R^i(M, M\oplus R)$$

$$\to \mathrm{Ext}_R^{i+1}(M/xM, M\oplus R) \to \cdots \qquad \text{for } \forall i\geq 0.$$

Hence $\mathrm{Ext}_R^{>0}(M, M\oplus R)=0 \Rightarrow \mathrm{Ext}_R^{i+1}(M/xM, M\oplus R)=0$ for $\forall i>0$

$$\mathrm{Ext}_R^{i+1}(M/xM, M\oplus R) \cong \mathrm{Ext}_{R/xR}^{i}(M/xM, M/xM\oplus R/xR)$$

Conversely,

$\mathrm{Ext}_{R/xR}^{>0}(M/xM, M/xM\oplus R/xR)=0$

$\Rightarrow \mathrm{Ext}_R^{\geq 1}(M/xM, M\oplus R)=0$

Hence $\mathrm{Ext}_R^i(M, M\oplus R) \xrightarrow{x} \mathrm{Ext}_R^i(M, M\oplus R) \to 0$ for $\forall i>0$.

i.e. $\mathrm{Ext}_R^i(M, M\oplus R)=0$ by NAK.

Thm (Fact (2)) Let $x \in \mathfrak{m}$ be a NZD of R.

Then $R : (\mathrm{ARC}) \iff R/xR : (\mathrm{ARC})$.

Proof) $\Leftarrow$: Let M be a f.g. R-mod. s.t. $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$

$$\overset{\text{lem (1)}}{\Rightarrow} \mathrm{Ext}_R^{>0}(\Omega, \Omega \oplus R) = 0$$

$$\overset{\text{lem (2)}}{\Rightarrow} \mathrm{Ext}_{R/xR}^{>0}(\Omega/x\Omega, \Omega/x\Omega \oplus R/xR) = 0$$

$$\overset{R/xR : (\mathrm{ARC})}{\Rightarrow} \Omega/x\Omega : R/xR\text{-free}$$

$$\Rightarrow \Omega : R\text{-free}$$

$$\Rightarrow M : R\text{-free}$$

$\Rightarrow$: N : f.g. R/xR-mod s.t. $\mathrm{Ext}_{R/xR}^{>0}(N, N \oplus R/xR) = 0$

In order to proceed in the same way as above, we face the following problem:

Problem Is there a f.g. R-mod M

s.t. ◦ $M/xM \simeq N$

◦ the min. R-free res. $F_\bullet \to M \to 0$ of M induces the min. R/xR-free res. of N:

$F_\bullet/xF_\bullet \to M/xM \simeq N \to 0$?

If yes, then

$0 \to M \xrightarrow{x} M \to N \to 0$ (ex) $\Big]\ \mathrm{Hom}_R(-, M \oplus R)$

$\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$ by NAK.

$\overset{R:\text{(ARC)}}{\rightsquigarrow}$ M : R-free

$\rightsquigarrow$ N : R/xR - free ☐

§3. Lifting problem

Let ◦ $R \to S$: ring hom. of Noeth. rings.

◦ N : f.g. S-mod.

Then $\quad N$ is liftable to R

$\overset{\text{def}}{\iff}$ $\exists M$: f.g. R-mod s.t.

$F_\bullet \to M \to 0$: R-free res. of M

induces an S-free res of N:

$$S \otimes F_\bullet \to S \otimes_R M \to 0 \ (\text{ex})$$
$$\begin{array}{c} \rotatebox{90}{$\cong$} \\ N \end{array}$$

$\iff$ $\exists M$: f.g. R-mod

s.t. $\begin{cases} S \otimes_R M \simeq N \text{ and} \\ \mathrm{Tor}^R_{>0}(S, M) = 0 \end{cases}$

Lifting problem

When is an S-mod N liftable to R?

Thm. [Auslander-Ding-Solberg].

Suppose
- $(R, \mathfrak{m})$: Noeth. local ring.
- $x \in \mathfrak{m}$: R-NZD.
- N : f.g. R/xR-mod.

If $\mathrm{Ext}^2_{R/xR}(N, N) = 0$, then N is liftable to R.

Set $R_i := R/x^iR$ for $i > 0$.

Prop Let N be a f.g. R_1-mod.

If $\exists \{L_i : R_i\text{-mod}\}_{i>0}$.

s.t. $\begin{cases} \cdot\ L_1 \simeq N \\ \cdot\ L_{i+1} \text{ is a lifting of } L_i, \end{cases}$

then N is liftable to R.

Philosophy)

$$\begin{tikzcd}
\cdots & L_3 \arrow[l, bend right] & L_2 \arrow[l, bend right] & L_1 = N \arrow[l, bend right] \\
\cdots \arrow[r, two heads] & R_3 \arrow[r, two heads] & R_2 \arrow[r, two heads] & R_1 \\
\cdots & R \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

$\rightsquigarrow$ The R-mod. $\varprojlim_i L_i$ is what we desired.

Proof of Thm)

Let $i \geq 1$. Suppose
- N : f.g. R_1-mod
- L_i : f.g. R_i-mod s.t. L_i is a lifting of N.

$$\boxed{\text{ETS: } L_i \text{ is liftable to } R_{i+1}}$$

Consider the min. R-free res. of L_i:

$$0 \to \Omega \to F \to L_i \to 0$$

$$\begin{tikzcd}[column sep=small]
\rightsquigarrow 0 \arrow[r] & (0):_{L_i} x \arrow[r] & \Omega/x\Omega \arrow[rr] \arrow[dr] & & F/xF \arrow[r] & N \arrow[r] & 0 \quad \text{ex.} \\
& & & \Omega^1_{R_1} N \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$(0):_{L_i} x = x^{i-1}L_i \cong L_i/(0):_{L_i} x^{i-1} = L_i/xL_i \cong N$$

These follow from that L_i is a lifting of N

On the other hand,

$$0 \to N \to \Omega/x\Omega \to \Omega^1_{R_1} N \to 0 \quad \text{ex}$$

$$\Updownarrow \ (1 \text{ to } 1)$$

$$\exists\ \theta \in \operatorname{Ext}^1_{R_1}(\Omega^1_{R_1} N, N) \cong \operatorname{Ext}^2_{R_1}(N, N)$$

$$\boxed{\text{Claim: } \theta = 0 \Rightarrow L_i \text{ is liftable to } R_{i+1}}$$

Proof) $\theta = 0 \iff 0 \to N \xrightarrow{f} \Omega/x\Omega \xrightarrow{g} \Omega_{R_1}' N \to 0$ splits.

$$\left(\text{i.e.}\quad \begin{tikzcd} N \arrow[d, equal] \arrow[r, "f"] & \Omega/x\Omega \arrow[dl, dashed, "\exists\beta"] \\ N & \end{tikzcd}\right)$$

$$\Rightarrow \begin{tikzcd}
0 \arrow[r] & \Omega \arrow[r] \arrow[d, "\varepsilon"] & F \arrow[r] \arrow[dd, "\exists"] & L_i \arrow[r] \arrow[dd, equal] & 0 \\
 & \Omega/x\Omega \arrow[d, "\beta"] \arrow[r, phantom, "\text{p.o.}"] & & & \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & L_i \arrow[r] & 0
\end{tikzcd}$$

Then E is a f.g. R_{i+1}-mod. and a lifting of L_i.

Proof)

$R_1 \otimes_R -$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, equal] & \Omega/x\Omega \arrow[r] \arrow[d, "\beta"] & F/xF \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & N \arrow[r, "t"'] & N \arrow[r] & E/xE \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Hence, since $\beta f = id_N$, t is surj

$\Rightarrow$ t is bij.

Hence $E/xE \simeq N$.

(To get $\mathrm{Tor}^{R_i}_{>0}(R_i, E) = 0$ is another story.) $\boxtimes$

Corollary

Let R be a complete intersection local ring and M a f.g. R-mod. Then

$$\mathrm{Ext}^2_R(M, M) = 0 \Rightarrow \mathrm{pd}_R M < \infty.$$

In particular, R satisfies (ARC).