Graded filtrations and Ideals of reduction number two

神代　真也

小山高専

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Introduction

Let

- $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring and
- I an $\mathfrak{m}$-primary ideal.

Then $\ell_A(A/I^{n+1})$ agrees with a polynomial function for $n \gg 0$, i.e. there exist integers $\mathrm{e}_0(I), \mathrm{e}_1(I), \ldots, \mathrm{e}_d(I)$ such that

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

for all $n \gg 0$.

Philosophy

Hilbert function $\ell_A(A/I^{n+1})$ reflects the structure of

- the **Rees algebra** $\mathcal{R}(I) = A[It] = \bigoplus_{n\geq 0} I^n t^n$ and
- the **associated graded ring** $\mathcal{G}(I) = \mathcal{R}(I)/I\mathcal{R}(I) = \bigoplus_{n\geq 0}(I^n/I^{n+1})t^n$.

Let

- $(A, \mathfrak{m})$ be a CM local ring of dimension $d \geq 2$,
- I an $\mathfrak{m}$-primary ideal, and
- $A/\mathfrak{m}$ an infinite field.

Choose a parameter reduction Q of I, i.e., $I^{n+1} = QI^n$ for some $n \geq 0$. Set the **reduction number** as

$$\mathrm{red}_Q I = \min\{n \geq 0 \mid I^{n+1} = QI^n\}.$$

Fact

- $\mathrm{red}_Q I = 0 \quad \Rightarrow \quad \mathcal{G}(I) \cong (A/I)[X_1, \ldots, X_d]$.
- In general, $\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I)$ holds, and "$=$" holds if and only if $\mathrm{red}_Q I = 1$.
  When this is the case, $\mathcal{G}(I)$ is a CM ring.

Question

$$\mathrm{red}_Q I = 2 \quad \Rightarrow \quad ???$$

Note that

- $\exists$ parameter reductions Q_1 and Q_2 of I such that $\mathrm{red}_{Q_1} I = 2$ and $\mathrm{red}_{Q_2} I = 3$.
- $\exists I$ with $\mathrm{red}_Q I = 2$ such that $\operatorname{depth} \mathcal{G}(I) = 0$.

(K, Israel J.)

$I^3 = QI^2$ and $\mathfrak{m} I^2 \subseteq QI \quad \Rightarrow \quad \ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$.
"$=$" holds if and only if $\operatorname{depth} \mathcal{G}(I) \geq d - 1$.

Preliminary (Sally module)

In what follows,

- $(A, \mathfrak{m})$ be a d-dimensional CM local ring,
- I an $\mathfrak{m}$-primary ideal, and
- $A/\mathfrak{m}$ an infinite field.

Choose a parameter reduction Q of I. Then, a f.g. graded

$\mathcal{R}(Q)$-module

$$S = I\mathcal{R}(I)/I\mathcal{R}(Q) = \bigoplus_{n \geq 0} (I^{n+1}/Q^n I)t^n$$

is called the **Sally module** of I w.r.t. Q.

Fact

- $\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - (\mathrm{e}_0(I) - \ell_A(A/I))\binom{n+d-1}{d-1} - \ell_A(S_n)$ for all $n \geq 0$.
- $\mathfrak{m}^\ell S = 0$ for $\ell \gg 0$.
- If $S \neq 0$, then $\mathrm{Ass}_{\mathcal{R}(Q)} S = \{\mathfrak{m}\mathcal{R}(Q)\}$.
- S is generated in degree 1 to $\mathrm{red}_Q I - 1$.

Problem

Give a nice filtration of S as a graded $\mathcal{R}(Q)/\mathfrak{m}^\ell\mathcal{R}(Q)$-module!

Main results

Key Theorem (K, Math. Nachr.)

Suppose that

- $R = \bigoplus_{n \geq 0} R_n$ is a standard graded Noetherian ring of dimension ≥ 2 such that $(R_0, \mathfrak{m}_0)$ is an Artinian local ring.
- $\mathfrak{m}_0 R$ is a prime ideal. (set $\mathfrak{p} = \mathfrak{m}_0 R$)
- $R/\mathfrak{p}$ satisfies Serre's condition (S_2).
- M is a f.g. graded R-module generated in single degree t such that $\operatorname{Ass}_R M = \{\mathfrak{p}\}$.

Then, the following assertions hold.

Key Theorem (K, Math. Nachr.) - continuation

- $\mathrm{e}_1(M) \leq t\mathrm{e}_0(M) + \ell_{R_\mathfrak{p}}(M_\mathfrak{p})\cdot\mathrm{e}_1(R/\mathfrak{p})$ holds.
- "$=$" holds if and only if there exists the following exact sequences.

$$\begin{aligned} 0 \to (R/\mathfrak{p})(-t) &\to M = M^0 \to M^1 \to 0, \\ 0 \to (R/\mathfrak{p})(-t) &\to M^1 \to M^2 \to 0, \\ &\vdots \\ 0 \to (R/\mathfrak{p})(-t) &\to M^{i_0-2} \to M^{i_0-1} \to 0, \text{ and} \\ 0 \to (R/\mathfrak{p})(-t) &\to M^{i_0-1} \to M^{i_0} = 0 \to 0 \end{aligned} \tag{1}$$

By applying Key Theorem to the Sally module as

$$R = \mathcal{R}(Q)/\mathfrak{m}^{\ell}\mathcal{R}(Q) \text{ and } M = \mathcal{R}(Q)S_{\mathrm{red}_Q I - 1},$$

we obtain the following:

Theorem A (K, Math. Nachr.)

If $r = \mathrm{red}_Q I \geq 2$, then the following are true:

- $\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \dfrac{\mathrm{e}_2(I)}{r-1}$.
- "$=$" if and only if $\operatorname{depth} \mathcal{G}(I) \geq d-1$.
  When this is the case, $r = 2$.

Further results

Theorem B (K, Math. Nachr.)

Suppose that I is integrally closed. Then,

- $\mathrm{red}_Q I = 2$ if and only if $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$.
- If $r = \mathrm{red}_Q I \geq 3$, then
$$\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \frac{(r-2)\ell_A(I^2/QI) + \mathrm{e}_2(I)}{r-1}.$$
- "$=$" if and only if depth $\mathcal{G}(I) \geq d-1$.
When this is the case, $r = 3$.

Example

Let $A = K[[X, Y]]$, $Q = (X^7, Y^7)$, and $I = Q + (X^6Y, X^5Y^2, X^2Y^5, XY^6)$. Then, $\mathrm{red}_Q I = 2$ and

$$\ell_A(A/I^{n+1}) = \begin{cases} 31 & (n = 0) \\ 49\binom{n+2}{2} - 21\binom{n+1}{1} & (n \geq 1). \end{cases}$$

It follows that $\ell_A(A/I) = 31 > \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I) = 28$; hence, $\operatorname{depth} \mathcal{G}(I) = 0$.

Example

Let $s > 0$, and let $A = K[[XY^i \mid 0 \leq i \leq 2s+1]]$, $Q = (X, XY^{2s+1})$, and $I = (XY^i \mid 0 \leq i \leq s) + (XY^{2s+1})$. Then, $I^2 \subseteq Q$, $\ell_A(I^2/QI) = s$, $\mathrm{red}_Q I = 2$ and

$$\ell_A(A/I^{n+1}) = (2s+1)\binom{n+2}{2} - 2s\binom{n+1}{1} + s$$

for all $n \geq 0$. It follows that $\operatorname{depth} \mathcal{G}(I) = 1$.

Method to give a filtration

Let

- R be a Noetherian ring and
- M a f.g. R-module such that $\mathrm{Ass}_R M = \{\mathfrak{p}\}$.

Then,

$$\exists 0 \to R/\mathfrak{p} \to M \to N^0 \to 0,$$

and

$$\exists 0 \to X^0 \to N^0 \to M^1 \to 0$$

such that $\mathrm{Ass}_R M^1 \subseteq \{\mathfrak{p}\}$ and $\mathrm{Ass}_R X^0 = \mathrm{Ass}_R N \setminus \{\mathfrak{p}\}$.

Method to give a filtration

- This process can be continued recursively.
- The argument also holds for the graded case.
- $X^i = 0$ for all i if and only if there exists the filtration such as (1).

Thank you for your attention.

References

- [S. Kumashiro], Graded filtrations and ideals of reduction number two, Mathematische Nachrichten, (to appear).
- [S. Kumashiro], Ideals of reduction number two, *Israel Journal of Mathematics*, **243**, 45–61, 2021