The Auslander-Reiten conjecture for non-Gorenstein Cohen-Macaulay rings

Conference on Commutative Algebra and its Interaction with Algebraic Geometry In Honor of Bernd Ulrich

Shinya Kumashiro (Chiba University, Japan)

1. Introduction

Auslander-Reiten conjecture (AR conjecture)

Let R be an Artin algebra and M a f.g. R-module. If

$$\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0,$$

then M is a projective R-module.

Fact (Araya, Huneke-Leuschke)

The AR conjecture holds for Gorenstein rings which are complete intersections in codimension one.

$\Rightarrow$ How about the AR conjecture for **non-Gorenstein rings**?

2. Main Results

Key Theorem

Suppose that R is a Gorenstein local ring. Let $Q = (x_1, x_2, \dots, x_n)$ be an ideal of R generated by a regular sequence on R. Then TFAE.

1. The AR conjecture holds for R.
2. $\exists \ell > 0$ s.t. the AR conjecture holds for R/Q^ℓ.
3. $1 \le \forall \ell \le n$, the AR conjecture holds for R/Q^ℓ.

This theorem provides two applications, i.e., Theorem A and Theorem B.

2-1. First application (determinantal rings)

Let $s \le t$ be positive integers. Set

- $A[\mathbf{X}] = A[X_{ij}]_{1\le i\le s, 1\le j\le t}$: poly. ring over a commutative ring A
- $\mathbb{I}_s(\mathbf{X})$: ideal of $A[\mathbf{X}]$ generated by the maximal minors of (X_{ij}).

With these notations, we have the following.

Theorem A (determinantal rings)

Suppose that

- $2s \le t+1$ and
- A is a Gorenstein ring which is a c.i. in codimension one.

Then the AR conjecture holds for the determinantal ring $A[\mathbf{X}]/\mathbb{I}_s(\mathbf{X})$.

2-2. Second application (Ulrich ideals)

Theorem B (Ulrich ideals)

Let $(R, \mathfrak{m})$ be a CM local ring. Suppose that

$\exists I$: Ulrich ideal s.t. R/I is a c.i.

Then the AR conjecture holds for R.

Here is the definition of Ulrich ideals.

Let $(R, \mathfrak{m})$ be a CM local ring and I an $\mathfrak{m}$-primary ideal. Then I is an *Ulrich ideal* if

- $I^2 = \mathfrak{q}I$ for some parameter ideal $\mathfrak{q}$ of R and
- I/I^2 is a free R/I-module.

3. Sketch of proofs

3-1. Proof of Theorem A

Proposition

Let $2s \le t+1$ be positive integers. Suppose that

- R is a Gorenstein local ring and
- $\{x_{ij}\}_{1\le i\le s, 1\le j\le t}$ forms a regular sequence on R.

Let I denote an ideal of R generated by maximal minors of (x_{ij}). Then

the AR conjecture holds for $R \Leftrightarrow$ it holds for R/I.

3-2. Proof of Theorem B

Proposition

With the assumption of Theorem B, we have the following.

- $d + r \le v$.
- $\exists S$: a c.i. local ring of dimension r and $\exists Q$: a parameter ideal of S s.t. $R/\mathfrak{q} \cong S/Q^2$.

$*$ d: dimension, r: CM type, v:embedding dimension.

3-3. Proof of Key Theorem

We focus on a proof of 1. $\Rightarrow$ 3. for the case where $\ell = 2$ and $n = \dim R$.

(1. $\Rightarrow$ 3.) Set $\overline{R} = R/Q^2$. Let M be a f.g. $\overline{R}$-module s.t. $\mathrm{Ext}_{\overline{R}}^{>0}(M, M \oplus \overline{R}) = 0$. Consider the exact sequence

$$0 \to Q/Q^2 \to \overline{R} \to R/Q \to 0$$

of $\overline{R}$-modules. Then $\mathrm{Ext}_{\overline{R}}^i(M, R/Q) \cong \mathrm{Ext}_{\overline{R}}^{i+1}(M, R/Q)^{\oplus n}$ for all $i > 0$. Hence $\mathrm{Ext}_{\overline{R}}^{\gg 0}(M, R/Q) = 0$, whence $\mathrm{Ext}_{\overline{R}}^{>0}(M, R/Q) = 0$. Thus

$$\mathrm{Tor}^{\overline{R}}_{>0}(M, R/Q) = 0$$

since R/Q is an Artinian Gorenstein ring. Hence we have the exact sequence

$$0 \to (M/QM)^{\oplus n} \to M \to M/QM \to 0$$

of $\overline{R}$-modules. Then, similarly to the above, $\mathrm{Ext}_{\overline{R}}^{>0}(M, M/QM) = 0$. Thus, since $\mathrm{Tor}^{\overline{R}}_{>0}(M, R/Q) = 0$, we have

$$\mathrm{Ext}_{R/Q}^{>0}(M/QM, M/QM \oplus R/Q) = 0.$$

Hence M/QM is a free R/Q-module since the AR conjecture holds for R/Q. This concludes that M is a free $\overline{R}$-module since $\mathrm{Tor}_1^{\overline{R}}(M, R/Q) = 0$. □

4. References

- [M. Auslander, I. Reiten], On a generalized version of the Nakayama conjecture, *Proceedings of the American Mathematical Society*, **52** (1975), 69–74.
- [S. Kumashiro], Auslander-Reiten conjecture for non-Gorenstein Cohen-Macaulay rings, arXiv:1906.02669.

This work was supported by JSPS KAKENHI Grant Number JP19J10579.