Ideals of reduction number two

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Introduction

Let

- $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring and
- I an $\mathfrak{m}$-primary ideal.

Then $\ell_A(A/I^{n+1})$ agrees with a polynomial function for $n \gg 0$, i.e. there exist integers $\mathrm{e}_0(I), \mathrm{e}_1(I), \ldots, \mathrm{e}_d(I)$ such that

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

for all $n \gg 0$.

Introduction

Philosophy

Hilbert function $\ell_A(A/I^{n+1})$ reflects the structure of

- the Rees algebra $\mathcal{R}(I) = A[It] = \bigoplus_{n \geq 0} I^n t^n$ and
- the associated graded ring $\mathcal{G}(I) = \mathcal{R}(I)/I\mathcal{R}(I) = \bigoplus_{n \geq 0} (I^n/I^{n+1}) t^n$.

Tools -1. reduction

Definition

An ideal J is called a reduction of I if

$$J \subseteq I \text{ and } I^{n+1} = JI^n \text{ for some } n \geq 0.$$

Fact

- If J is a reduction of I, then $\mathrm{e}_0(I) = \mathrm{e}_0(J)$.
- Suppose that $A/\mathfrak{m}$ is infinite. Then, for any $\mathfrak{m}$-primary ideal I, there exists a parameter ideal Q such that Q is a reduction of I.

Hence, if $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring with the infinite residue field $A/\mathfrak{m}$, then $\mathrm{e}_0(I) = \mathrm{e}_0(Q) = \ell_A(A/Q)$ for some parameter reduction Q.

Next target: $\mathrm{e}_1(I)$

Tools -1. reduction

Example

Let K be a field.

- Set $A = K[[X, Y]]$, $I = (X^5, X^3Y^2, X^2Y^3, Y^5)$, and $Q = (X^5, Y^5)$. Then $I^3 = QI^2$ and

$$\ell_A(A/I^{n+1}) = \begin{cases} 17 & (n = 0) \\ 25\binom{n+2}{2} - 10\binom{n+1}{1} + 2 & (n \geq 1). \end{cases}$$

- Let $A = K[[\mathbb{X}]]/I_s(\mathbb{X})$ and $\mathfrak{m}$ the maximal ideal, where $\mathbb{X} = (X_{ij})$ is an $s \times t$ matrix with st variables. Then $\mathfrak{m}^s = Q\mathfrak{m}^{s-1}$ for a suitable parameter ideal Q.

Tools -2. Sally module

In what follows,

- $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring,
- I an $\mathfrak{m}$-primary ideal, and
- Q a parameter reduction of I.

Then a f.g. graded $\mathcal{R}(Q)$-module

$$\mathcal{S}_Q(I) = I\mathcal{R}(I)/I\mathcal{R}(Q) = \bigoplus_{n \geq 0} (I^{n+1}/Q^n I)t^n$$

is called the Sally module of I w.r.t. Q.

Tools -2. Sally module

Fact

- For all integer $n \geq 0$, we have
$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - (\mathrm{e}_0(I) - \ell_A(A/I))\binom{n+d-1}{d-1} - \ell_A([\mathcal{S}_Q(I)]_n).$$
- If $\mathcal{S}_Q(I) \neq 0$, then $\mathrm{Ass}_{\mathcal{R}(Q)} \mathcal{S}_Q(I) = \{\mathfrak{m}\mathcal{R}(Q)\}$.

Known results

Let

- $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring,
- I an $\mathfrak{m}$-primary ideal, and
- Q a parameter reduction of I.

Suppose $d = \dim A \geq 1$. Then

- (Northcott) $\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I)$.
- (Huneke, Ooishi) $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) \Leftrightarrow I^2 = QI$. When this is the case, $\mathcal{G}(I)$ is a CM ring, and so is $\mathcal{R}(I)$ if $d \geq 2$.

Known results

- (Sally) $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) + 1$ and $\mathrm{e}_2(I) \neq 0$
  $\Leftrightarrow I^3 = QI^2$ and $\ell_A(I^2/QI) = 1$
  $\Rightarrow \operatorname{depth} \mathcal{G}(I) \geq d - 1$.
- (Goto-Nishida-Ozeki) $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) + 1$
  $\Leftrightarrow \mathcal{S}_Q(I) \cong (X_1, \ldots, X_c) \subseteq (A/\mathfrak{m})[X_1, \ldots, X_d]$,
  where $c = \ell_A(I^2/QI)$
  $\Rightarrow I^3 = QI^2$, $\mathfrak{m}I^2 \subseteq QI$, and $\operatorname{depth} \mathcal{G}(I) \geq d - c$.

Question

- $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) + 2 \Rightarrow$???
- $I^3 = QI^2 \Rightarrow$???

Note that reduction number depends on the choice of Q in general. Hence $I^3 = QI^2$ may not be characterized by the Hilbert coefficients.

Main Results

Suppose $d \geq 2$.

Main Theorem [K]

Suppose that $I^3 = QI^2$ and $\mathfrak{m}I^2 \subseteq QI$. Then

$$\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I).$$

When this is the case, "$=$" holds $\Leftrightarrow$ $\operatorname{depth} \mathcal{G}(I) \geq d - 1$.

Remark

- (Narita) $\mathrm{e}_2(I) \geq 0$.
- (Corso-Polini-Rossi) If I is integrally closed, then $\ell_A(A/I) \leq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$.

Main Results

Suppose $d \geq 2$.

Corollary (cf. Corso-Polini-Rossi)

Suppose that $\mathfrak{m}^3 = Q\mathfrak{m}^2$. Then

$$1 = \ell_A(A/\mathfrak{m}) = \mathrm{e}_0(\mathfrak{m}) - \mathrm{e}_1(\mathfrak{m}) + \mathrm{e}_2(\mathfrak{m}).$$

When this is the case, $\operatorname{depth} \mathcal{G}(\mathfrak{m}) \geq d - 1$.

Examples

Example

- Set $A = K[[X, Y]]$, $I = (X^5, X^3Y^2, X^2Y^3, Y^5)$, and $Q = (X^5, Y^5)$. Then $I^3 = QI^2$, $\mathfrak{m}I^2 \subseteq QI$, and

$$\ell_A(A/I^{n+1}) = \begin{cases} 17 & (n = 0) \\ 25\binom{n+2}{2} - 10\binom{n+1}{1} + 2 & (n \geq 1). \end{cases}$$

  It follows that $\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I) = 17$, whence $\operatorname{depth} \mathcal{G}(I) \geq 1$.

- Let $A = K[[\mathbb{X}]]/I_3(\mathbb{X})$ and $\mathfrak{m}$ the maximal ideal. Then $\mathfrak{m}^3 = Q\mathfrak{m}^2$ for a suitable parameter ideal Q. It follows that $1 = \mathrm{e}_0(\mathfrak{m}) - \mathrm{e}_1(\mathfrak{m}) + \mathrm{e}_2(\mathfrak{m})$ and $\operatorname{depth} \mathcal{G}(\mathfrak{m}) \geq \dim A - 1 = 2t + 1$.

Examples

Example

Set $A = K[[X, Y]]$, $I = (X^7, X^6Y, X^5Y^2, X^2Y^5, XY^6, Y^7)$, and $Q = (X^7, Y^7)$. Then $I^3 = QI^2$, $\mathfrak{m}I^2 \not\subseteq QI$, and

$$\ell_A(A/I^{n+1}) = \begin{cases} 31 & (n = 0) \\ 49\binom{n+2}{2} - 21\binom{n+1}{1} & (n \geq 1). \end{cases}$$

Hence $\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$ holds.

Proof of Main Theorem

Set

$$R = \mathcal{R}(I),\ T = \mathcal{R}(Q), \text{ and } S = \mathcal{S}_Q(I).$$

We may assume that $S \neq 0$.

Claim

The following are equivalent:

- $I^3 = QI^2$ and $\mathfrak{m}I^2 \subseteq QI$;
- $S = TS_1$ and $\mathfrak{m}S = 0$.

Hence, S is a f.g. graded $T/\mathfrak{m}T$-module. Set $P = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, \ldots, X_d]$. Then we get $\mathrm{Ass}_P\, S = \{0\}$ since $\mathrm{Ass}_T\, S = \{\mathfrak{m}T\}$.

Proof of Main Theorem

Theorem (graded Bourbaki sequence)

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian normal domain such that R_0 is an infinite field. Let $M = RM_0$ be a torsionfree graded R-module of rank $r > 0$. Then there exist a graded ideal J and $m \in \mathbb{Z}$ such that

$$0 \to R^{r-1} \to M \to J(m) \to 0$$

is a graded exact sequence.

By applying the theorem, we get the exact sequence

$$0 \to P(-1)^{r-1} \to S \to J(m) \to 0,$$

where J is a graded ideal of P and $m \in \mathbb{Z}$.

Proof of Main Theorem

That is,

$$0 \to P(-1)^{r-1} \to S \to P(m) \to (P/J)(m) \to 0.$$

We may assume that $\mathrm{ht}_P J \geq 2$. Then

$$\begin{aligned}
\ell_A(S_n) =& (r-1)\ell_A(P_{n-1}) + \ell_A(P_{n+m}) - \ell_A((P/J)_{n+m}) \\
=& (r-1)\binom{n-1+d-1}{d-1} + \binom{n+m+d-1}{d-1} - \text{(lower terms)} \\
=& r\binom{n+d-1}{d-1} - (r-1-m)\binom{n+d-2}{d-2} - \text{(lower terms)}.
\end{aligned}$$

Proof of Main Theorem

Therefore,

$$\begin{aligned}
&\ell_A(A/I^{n+1})\\
=&\mathrm{e}_0(I)\binom{n+d}{d} - (\mathrm{e}_0(I) - \ell_A(A/I))\binom{n+d-1}{d-1} - \ell_A(S_n)\\
=&\mathrm{e}_0(I)\binom{n+d}{d} - (\mathrm{e}_0(I) - \ell_A(A/I) + r)\binom{n+d-1}{d-1}\\
&+ (r-1-m)\binom{n+d-2}{d-2} + \text{(lower terms)}.
\end{aligned}$$

Hence we obtain

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + r \quad \text{and} \quad \mathrm{e}_2(I) = r - 1 - m.$$

Proof of Main Theorem

It follows that

$$\ell_A(A/I) = \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I) + 1 + m.$$

On the other hand, since $S \to J(m) \to 0$ is exact, J is generated by elements of degree $m+1$. Hence $m+1 \geq 0$. Thus we have $\ell_A(A/I) \geq \mathrm{e}_0(I) - \mathrm{e}_1(I) + \mathrm{e}_2(I)$ as desired.

Thank you for your attention.