The Auslander-Reiten conjecture for certain non-Gorenstein CM rings

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Introduction

In this talk, let

- R be a commutative Noetherian ring and
- M a finitely generated R-module.

Conjecture (Auslander-Reiten, 1975)

(ARC): $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0 \Rightarrow M$ is projective.

Known results

(ARC) holds for R if R satisfies one of the followings.

- Gorenstein normal domain. [Araya]
- complete intersection. [Auslander-Ding-Solberg]
- Gorenstein s.t. $\mathrm{e}(R) - \dim R \leq 4$. [Şega]
- CM normal excellent $\mathbb{Q}$-algebra. [Huneke-Leuschke]

$\vdots$

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $Q = (x_1, x_2, \ldots, x_n)$ an ideal of R generated by a regular sequence on R.

Fact [Auslander-Ding-Solberg]

R satisfies **(ARC)** $\Leftrightarrow$ R/Q satisfies **(ARC)**.

Question

R satisfies **(ARC)** $\Leftrightarrow$ R/Q^ℓ satisfies **(ARC)**?

Motivations

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $Q = (x_1, x_2, \ldots, x_n)$ an ideal of R generated by a regular sequence on R. Let $\ell > 1$. Then

- R/Q^ℓ is neither a Gorenstein ring nor a domain.
- $Q^\ell = \mathbb{I}_\ell \begin{pmatrix} x_1 & \cdots & \cdots & x_n & & 0 \\ & \ddots & & & \ddots & \\ 0 & & x_1 & \cdots & \cdots & x_n \end{pmatrix}$.
- $R[Qt] \cong R[Y_1, \ldots, Y_n]/\mathbb{I}_2\left(\begin{smallmatrix} x_1 & \cdots & \cdots & x_n \\ Y_1 & \cdots & \cdots & Y_n \end{smallmatrix} \right)$.
- $R[t^a, t^b, t^c] \cong \begin{cases} R[X, Y, Z]/(\text{reg. seq. of } R[X, Y, Z]) \\ R[X, Y, Z]/\mathbb{I}_2\left(\begin{smallmatrix} X^\alpha & Y^\beta & Z^\gamma \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{smallmatrix} \right) \end{cases}$
- If R has minimal multiplicity, then $\exists Q$ s.t. $R/Q \cong S/\mathfrak{n}^2$ where $(S, \mathfrak{n})$ is a RLR.

Main results

$Q = (x_1, x_2, \ldots, x_n)$ an ideal of R generated by a regular sequence on R. Let $\ell > 0$.

Key proposition [K]

Suppose that R is Gorenstein. Consider the followings.

1. R satisfies **(ARC)**.
2. R/Q^ℓ satisfies **(ARC)**.

Then (2) $\Rightarrow$ (1) holds. (1) $\Rightarrow$ (2) also holds if $\ell \leq n$.

In particular,

$$R \text{ satisfies } \textbf{(ARC)} \Leftrightarrow R/Q^2 \text{ satisfies } \textbf{(ARC)}$$

since we may assume that $n \geq 2$.

Main results

Let $s \leq t$ be positive integers and A a commutative ring. Set

- $A[\mathrm{X}] = A[X_{ij}]_{1 \leq i \leq s, 1 \leq j \leq t}$ **:** polynomial ring over A
- $\mathbb{I}_s(\mathrm{X})$ **:** ideal of $A[\mathrm{X}]$ generated by the maximal minors of the matrix (X_{ij}).

With these notations, we have the following.

Theorem A

Suppose A is either a complete intersection or a Gorenstein normal domain. Then **(ARC)** holds for the determinantal ring $A[\mathrm{X}]/\mathbb{I}_s(\mathrm{X})$ if $2s \leq t+1$.

Main results

Suppose that

- $(A, \mathfrak{n})$: Gorenstein normal domain or complete intersection
- $Q = (x_1, x_2, \ldots, x_d)$: parameter ideal of A.

$*$ d: dimension of A.

Corollary

(ARC) holds for the following rings:

- the Rees algebra $A[[Qt]]$;
- three generated numerical semigroup ring $A[[t^a, t^b, t^c]]$;
- ring having minimal multiplicity. (it is known)

Main results

Let $(R, \mathfrak{m})$ be a CM local ring and I an $\mathfrak{m}$-primary ideal. Then I is an *Ulrich ideal* if

- $I^2 = \mathfrak{q}I$ for some parameter ideal $\mathfrak{q} \subseteq I$ and
- I/I^2 is a free R/I-module.

R has minimal multiplicity $\Leftrightarrow$ $\mathfrak{m}$ is an Ulrich ideal.

Theorem B

Let R be a CM local ring. Suppose that

$\exists I$**:** Ulrich ideal s.t. R/I is a complete intersection.

Then **(ARC)** holds for R.

Main results

Example

Let

- $R = k[[t^6, t^{11}, t^{16}, t^{26}]]$.
- $I = (t^6, t^{16}, t^{26})$: an ideal of R.

Then I is an Ulrich ideal and R/I is a complete intersection.

Note that

$$R \cong T/\left(X^7 - ZW, Y^2 - XZ, Z^2 - XW, W^2 - X^6 Z\right),$$

where $T = k[[X, Y, Z, W]]$. The kernel does not form a determinantal ideal.

Sketch of Key proposition

We may assume that Q is a parameter ideal.
(1) $\Rightarrow$ (2): Suppose that R satisfies **(ARC)**. Let N be a f.g. R/Q^ℓ-module s.t.

$$\operatorname{Ext}_{R/Q^\ell}^{>0}(N, N \oplus R/Q^\ell) = 0.$$

The goal is that N is a free R/Q^ℓ-module.
We analyze the following exact sequences:

$$\begin{array}{c} 0 \to Q/Q^2 \to R/Q^2 \to R/Q \to 0 \\ \vdots \\ 0 \to Q^{\ell-2}/Q^{\ell-1} \to R/Q^{\ell-1} \to R/Q^{\ell-2} \to 0 \\ 0 \to Q^{\ell-1}/Q^\ell \to R/Q^\ell \to R/Q^{\ell-1} \to 0. \end{array} \tag{1}$$

Sketch of Key proposition

By applying $\mathrm{Hom}_{R/Q^\ell}(N, -)$ to (1), we get long exact sequences and an isomorphism

- $$\cdots \to \mathrm{Ext}^j_{R_\ell}(N, R_1)^n \to \mathrm{Ext}^j_{R_\ell}(N, R_2) \to \mathrm{Ext}^j_{R_\ell}(N, R_1) \to \mathrm{Ext}^{j+1}_{R_\ell}(N, R_1)^n \to \cdots$$

  $\vdots$

- $$\cdots \to \mathrm{Ext}^j_{R_\ell}(N, R_1)^{\binom{\ell+n-3}{n-1}} \to \mathrm{Ext}^j_{R_\ell}(N, R_{\ell-1}) \to \mathrm{Ext}^j_{R_\ell}(N, R_{\ell-2}) \to \mathrm{Ext}^{j+1}_{R_\ell}(N, R_1)^{\binom{\ell+n-3}{n-1}} \to \cdots$$

- $$\mathrm{Ext}^j_{R_\ell}(N, R_{\ell-1}) \cong \mathrm{Ext}^{j+1}_{R_\ell}(N, R_1)^{\binom{\ell+n-2}{n-1}}$$

for all $j > 0$. Here, R_i denotes R/Q^i.

Sketch of Key proposition

Set $E_j = \ell_{R_\ell}(\operatorname{Ext}^j_{R_\ell}(N, R_1))$ for $j > 0$. Then we obtain that

$$\begin{aligned}
\binom{\ell+n-2}{n-1} E_{j+1} &= \ell_{R_\ell}(\operatorname{Ext}^j_{R_\ell}(N, R_{\ell-1})) \\
&\leq \binom{\ell+n-3}{n-1} E_j + \ell_{R_\ell}(\operatorname{Ext}^j_{R_\ell}(N, R_{\ell-2})) \\
&\vdots \\
&\leq \left(\sum_{i=0}^{\ell-2} \binom{i-1+n}{n-1} \right) E_j = \binom{\ell+n-2}{n} E_j.
\end{aligned}$$

Hence $E_{j+1} \leq \dfrac{\ell-1}{n} E_j$ for all $j > 0$. It follows that $E_{j+1} = 0$ for all $j \gg 0$ since $\ell \leq n$.

Sketch of Key proposition

On the other hand, we get that

$$E_{j+1} = 0 \Rightarrow E_j = 0$$

for all $j > 0$ by the long exact sequences and an isomorphism. Therefore, $\mathrm{Ext}^{>0}_{R_\ell}(N, R_1) = 0$. Since R_1 is an Artinian Gorenstein ring, it follows that $\mathrm{Tor}^{R_\ell}_{>0}(N, R_1) = 0$, that is N is a lifting of N/QN.
Hence, we have similar short exact sequences to (1) for N/Q^iN. It follows that $\mathrm{Ext}^{>0}_{R_\ell}(N, N/QN) = 0$ and thus $\mathrm{Ext}^{>0}_{R_1}(N/QN, N/QN) = 0$.
Therefore, since R_1 satisfies (ARC), N/QN is a free R_1-module. Hence N is a free R_ℓ-module.

Thank you for your attention.