Residually faithful modules and the Cohen-Macaulay type of idealizations

S. Kumashiro

Chiba University

j.w.w. S. Goto and N. T. H. Loan

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Introduction

Let R be a commutative ring and
M an R-module.

We set $A = R \oplus M$ as an additive group and define the multiplication in A by

$$(a, x) \cdot (b, y) = (ab, ay + bx)$$

for $(a, x), (b, y) \in A$. Then, A forms a commutative ring, which we denote by $A = R \ltimes M$ and call the idealization of M over R.

Fact

$R \ltimes M$		R		M
Noether	$\Leftrightarrow$	Noether	$+$	fin. gene.
CM	$\Leftrightarrow$	CM	$+$	MCM
Gorenstein	$\Leftrightarrow$	CM	$+$	canon. module

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Problem 1.

Find "nice" MCM modules of R by analyzing the Cohen-Macaulay type of the idealization $R \ltimes M$.

Setting

Suppose

- $(R, \mathfrak{m})$: a Cohen-Macaulay local ring of dimension d
- M:a non-zero maximal Cohen-Macaulay R-module
- $A = R \ltimes M$

$$\mathrm{r}_R(M) = \ell_R(\mathrm{Ext}_R^d(R/\mathfrak{m}, M))$$

denotes the Cohen-Macaulay type of M. Set $\mathrm{r}(R) = \mathrm{r}_R(R)$.

Let $\mathrm{Q}(R)$ be the total ring of fractions of R.
For R-submodules X and Y of $\mathrm{Q}(R)$, let

$$X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}$$

and $X :_R Y = (X : Y) \cap R$.

Outline

1. The inequality $r_R(M) \leq r(R \ltimes M) \leq r_R(M) + r(R)$
2. Condition $r_R(M) = r(R \ltimes M)$
3. Condition $r(R \ltimes M) = r_R(M) + r(R)$
4. Other related topics

Outline

Lemma 2.

Let $(R, \mathfrak{m})$ be a commutative local ring and let M be an R-module. We set $A = R \ltimes M$ and denote by $\mathfrak{n} = \mathfrak{m} \times M$ the maximal ideal of A. Then

$$(0) :_A \mathfrak{n} = ([(0) :_R \mathfrak{m}] \cap \operatorname{Ann}_R M) \times [(0) :_M \mathfrak{m}] .$$

Therefore, when R is an Artinian local ring,

$$(0) :_A \mathfrak{n} = (0) \times [(0) :_M \mathfrak{m}] \Leftrightarrow \operatorname{Ann}_R M = (0).$$

Theorem 3.

Let R be a CM local ring and
M an MCM R-module. Set $A = R \ltimes M$. Then

$$\mathrm{r}_R(M) \le \mathrm{r}(A) \le \mathrm{r}_R(M) + \mathrm{r}(R).$$

Let $\mathfrak{q}$ be a parameter ideal of R. We then have the following.

(1) $\mathrm{r}(A) = \mathrm{r}_R(M) \Leftrightarrow M/\mathfrak{q}M$ *is a faithful $R/\mathfrak{q}$-module.*
(2) $\mathrm{r}(A) = \mathrm{r}_R(M) + \mathrm{r}(R) \Leftrightarrow (\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$.

(proof) We may assume that R is Artin. Then, since

$$(0) \subseteq [(0) :_R \mathfrak{m}] \cap \mathrm{Ann}_R\, M \subseteq (0) :_R \mathfrak{m},$$

$$\mathrm{r}_R(M) \le \mathrm{r}(A) \le \mathrm{r}_R(M) + \mathrm{r}(R).$$

Example 4.

Let k be a field and set

$$S = k[[X_1, X_2, \ldots, X_\ell]] \ (\ell \geq 2).$$

Suppose that $R = S/\mathbb{I}_2(\mathbb{M})$, where $\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \ldots & X_{\ell-1} & X_\ell \\ X_2 & X_3 & \ldots & X_\ell & X_1^2 \end{pmatrix}$.
Then R is a one-dimensional CM local ring with $\mathrm{r}(R) = \ell - 1$.
Consider ideals

$$I_i = (x_1) + (x_i, x_{i+1}, \ldots, x_\ell) \quad \text{for all } 2 \leq i \leq \ell$$

of R, where x_j denotes the image of X_j in R. Then

$$\mathrm{r}(R \ltimes I_i) = \mathrm{r}_R(I_i) + (\ell - i + 1).$$

Proposition 5.

Let $M \in \Omega\mathrm{CM}(R)$. Then

$$\mathrm{r}(R \ltimes M) = \begin{cases} \mathrm{r}_R(M) & \text{if } R \text{ is a direct summand of } M, \\ \mathrm{r}_R(M) + \mathrm{r}(R) & \text{otherwise.} \end{cases}$$

(proof) Take an exact sequence

$$0 \to M \xrightarrow{\varphi} F \to X \to 0,$$

where F is free and X is an MCM R-module. Let $\mathfrak{q}$ be a parameter ideal of R. Then

$$0 \to M/\mathfrak{q}M \xrightarrow{\overline{\varphi}} F/\mathfrak{q}F \to X/\mathfrak{q}X \to 0.$$

If M has no free summands, $\overline{\varphi}(M/\mathfrak{q}M) \subseteq \mathfrak{m}(F/\mathfrak{q}F)$. Hence, $(\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$.

Outline

Definition 6 (Brennan-Vasconcelos (2001)).

Let M be an MCM R-module. We say that M is residually faithful, if $M/\mathfrak{q}M$ is a faithful $R/\mathfrak{q}$-module for some parameter ideal $\mathfrak{q}$ of R.

Note that

$$M \text{ is a residually faithful module} \Leftrightarrow \mathrm{r}_R(M) = \mathrm{r}(R \ltimes M).$$

Therefore, the property of residually faithful is independent of the choice of a parameter ideal $\mathfrak{q}$ of R.

Theorem 7.

Suppose that R possesses the canonical module K_R. Set

$$t : \mathrm{Hom}_R(M, \mathrm{K}_R) \otimes_R M \to \mathrm{K}_R, \text{ where } t(f \otimes x) = f(x).$$

Then

$$\mathrm{r}(A) = \mathrm{r}_R(M) + \mu_R(\mathrm{Coker}\, t).$$

Corollary 8.

Suppose that R possesses the canonical module K_R. Then TFAE.

(1) $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$.

(2) $t : \mathrm{Hom}_R(M, \mathrm{K}_R) \otimes_R M \to \mathrm{K}_R$ *is surjective.*

(3) *M is a residually faithful R-module.*

Proposition 9 (Properties of residaully faithful (RF)).

Let M be an MCM R-module. Then the following assertions hold true.

(1) *Let $a \in \mathfrak{m}$ be a NZD of R and $\overline{*} = R/aR \otimes_R *$. Then*
M *is a* (RF) R*-module* $\Leftrightarrow$ $\overline{M}$ *is a* (RF) $\overline{R}$*-module.*

(2) *Suppose that $\exists \mathrm{K}_R$. Then*
M *is a* (RF) R*-module* $\Leftrightarrow$ $M^{\vee}$ *is a* (RF) R*-module.*

(3) *Let $\varphi : R \to S$ be a flat local homomorphism of CM local rings. Then*
M *is a* (RF) R*-module* $\Leftrightarrow$ $S \otimes_R M$ *is a* (RF) S*-module.*

(4) *Suppose that M is a (RF) R-module. Then*
- *M is a faithful R-module.*
- *$M_{\mathfrak{p}}$ is a (RF) $R_{\mathfrak{p}}$-module for $\forall \mathfrak{p} \in \operatorname{Spec} R$.*

Outline

Notation

Set

- $\mathrm{CM}(R) = \{M \mid M \text{ is a nonzero MCM } R\text{-module}\}$
- $\Omega\mathrm{CM}(R) = \{M \in \mathrm{CM}(R) \mid \exists 0 \to M \to F \to X \to 0,$ s.t. F is free and $X \in \mathrm{CM}(R)\}$
- $\Omega\mathrm{CM}^{\times}(R) = \{M \in \Omega\mathrm{CM}(R) \mid M \text{ has no free summands}\}$
- $\mathrm{Ul}(R) = \{M \in \mathrm{CM}(R) \mid \mu_R(M) = \mathrm{e}^0_{\mathfrak{m}}(M)\}$

Note that $M \in \mathrm{Ul}(R)$ is called an Ulrich R-module.

Set

$$\mathcal{A} = \{M \in \mathrm{CM}(R) \mid \mathrm{r}(R \ltimes M) = \mathrm{r}_R(M) + \mathrm{r}(R)\}.$$

Theorem 10.

The inclusions

$$\Omega\mathrm{CM}^{\times}(R) \subseteq \mathcal{A} \text{ and } \mathrm{UI}(R) \subseteq \mathcal{A}$$

holds and we have the following.

(1) $\mathrm{UI}(R) = \mathcal{A} \Leftrightarrow R$ *has maximal embedding dimension.*

(2) [Kobayashi (2017)] *Suppose that* $\dim R = 1$ *and* $\mathrm{Q}(\hat{R})$ *is a Gorenstein ring. Then* $\Omega\mathrm{CM}^{\times}(R) = \mathcal{A} \Leftrightarrow R$ *is an almost Gorenstein local ring.*

Outline

Bounding the supremum $\sup \mathrm{r}(R \ltimes M)$

Let $r > 0$ be an integer and set

$$\mathcal{F}_r(R) = \{M \in \mathrm{CM}(R) \mid M \subseteq R^{\oplus r}\}.$$

Theorem 11.

Let $M \in \mathcal{F}_r(R)$. Then

$$\mathrm{r}(R \ltimes M) \le \mathrm{r}(R) + r \cdot \mathrm{e}(R).$$

The equality holds if and only if either R is a RLR or M is an Ulrich R-module, possessing rank r.

Bounding the supremum $\sup r(R \ltimes M)$

Corollary 12.

Suppose that $(R, \mathfrak{m})$ is a CM local ring of dimension one. Let $\mathcal{F}$ be the set of $\mathfrak{m}$-primary ideals of R. Then

$$\sup_{I \in \mathcal{F}} r(R \ltimes I) = \begin{cases} 1 & \text{if } R \text{ is a DVR,} \\ r(R) + e(R) & \text{otherwise.} \end{cases}$$

One-dimensional case

Assume that $\dim R = 1$. Let I be an $\mathfrak{m}$-primary ideal.

Definition 13.

(1) I is called a closed ideal if $I : I = R$.

(2) I is called a trace ideal if $(R : I)I = I$.

One-dimensional case

Theorem 14.

Suppose that R is a Gorenstein ring and let I be an $\mathfrak{m}$-primary ideal. Then the following assertions hold true.

(1) *Every closed ideal of R is principal.*

(2) $r(R \ltimes I) = 1 + r_R(I)$, *if* $\mu_R(I) > 1$.

(3) $r(R/I) \le r_R(I) \le 1 + r(R/I)$.

(4) *If I is a trace ideal, then* $r_R(I) = 1 + r(R/I)$.

Corollary 15.

Let R be a Gorenstein ring which is not a DVR. *Then $R \ltimes \mathfrak{m}$ is an almost Gorenstein ring, possessing* $r(R \ltimes \mathfrak{m}) = 3$.

Thank you for your attention.